THE p-ADIC COMPLEX NUMBERS

YIHANG ZHU

1. Basic properties

The residue field of $\mathbb{Q}_p$ is $\mathbb{F}_p$, which is not algebraically closed. Therefore $\mathbb{Q}_p$ is not algebraically closed. We extend the p-adic valuation and absolute value on $\mathbb{Q}_p$ to $\bar{\mathbb{Q}}_p$, denoted by $|\cdot|$ and v. Note that v on $\bar{\mathbb{Q}}_p$ is no longer discrete. By definition, we have $v(x) = [L : \mathbb{Q}_p]^{-1} v(\mathrm{N}_{L/\mathbb{Q}_p} x)$, if $x \in L$ with $L/\mathbb{Q}_p$ a finite extension. We normalize so that $v(p) = 1, |p| = p^{-1}$.

Lemma 1.1. *Let m be a positive integer coprime to p. The m-th roots of unity $\{\zeta_i, 1 \le i \le m\}$ in $\bar{\mathbb{Q}}_p$ are pairwise non-congruent modulo v i.e. $v(\zeta_i - \zeta_j) = 0, i \neq j$.*

Proof. Suppose $\{\zeta_i, 1 \le i \le m-1\}$ are the m-th roots of unity apart from 1. Then $\prod_{1 \le i \le m-1}(1-\zeta_i) = \frac{X^m - 1}{X-1}|_{X=1} = m$. But $v(m) = 0$, so $v(1-\zeta_i) = 0$. □

Proposition 1.2. *$\bar{\mathbb{Q}}_p$ is not complete.*

Proof. Suppose $\bar{\mathbb{Q}}_p$ is complete. Then the following series should converge to an element $\alpha \in \bar{\mathbb{Q}}_p$.

$$\alpha = \sum_{n=1}^{\infty} \zeta_n p^n, \tag{1}$$

where ζ_n is a primitive n-th root of unity in $\bar{\mathbb{Q}}_p$ if $p \nmid n$, and $\zeta_n := 1$ if $p | n$. Let $K/\mathbb{Q}_p$ be a finite extension such that $\alpha \in K$. We prove by induction that K contains all the ζ_n's. But then since the residue field of K is finite, we have a contradiction, by the previous Lemma.

To show that K contains all the ζ_n, suppose $p \nmid m$ and K contains ζ_n for $n < m$. Then K contains the element

$$\beta = p^{-m}(\alpha - \sum_{n<m} \zeta_n p^n). \tag{2}$$

But $\beta \equiv \zeta_m \mod p$, so by Hensel's lemma, the element $\zeta_m \mod p$, which is contained in the residue field of K, lifts to an m-th root of unity in K. But the latter has to be ζ_m itself by the previous Lemma. □

We let $\mathbb{C}_p$ be the p-adic completion of $\bar{\mathbb{Q}}_p$, called the field of p-adic complex numbers.

Proposition 1.3. *$\mathbb{C}_p$ is algebraically closed.*

Proof. Let $f(x) \in \mathbb{C}_p[x]$, we need to show $\mathbb{C}_p$ contains a root of $f(x)$. Without loss of generality, we may assume $f(x)$ is monic with coefficients in $\mathcal{O} = \mathcal{O}_{\mathbb{C}_p}$. We pick a sequence of monic polynomials $f_n(X) \in \mathcal{O}_{\bar{\mathbb{Q}}_p}[x]$ that coefficient-wise converge to $f(X)$. Say $f_{n+1} - f_n$ has coefficients in $\{v \ge N_n\}, N_n \to \infty$. Let α_n

be a root of $f_n(X)$ in $\bar{\mathbb{Q}}_p$. Then necessarily $v(\alpha_n) \geq 0$. We have $v(f_{n+1}(\alpha_n)) = v(f_{n+1}(\alpha_n) - f_n(\alpha_n)) \geq N_n \to \infty$. But if $f_{n+1}(X) = \prod_i (X - \beta_i)$, then

$$v(f_{n+1}(\alpha_n)) = \sum_i v(\alpha_n - \beta_i) \geq N_n,$$

so f_{n+1} has a root α_{n+1} with $v(\alpha_{n+1} - \alpha_n) \geq N_n / \deg f$. Thus we get a Cauchy sequence $\{\alpha_n\}_n \subset \bar{\mathbb{Q}}_p$, whose limit in $\mathbb{C}_p$ is a root of $f(X)$.

Alternatively, let α be a root of $f(X)$ in $\bar{\mathbb{C}}_p$. Find a monic polynomial $g(X) \in \mathcal{O}_{\bar{\mathbb{Q}}_p}[X]$ that is coefficient-wise close to $f(X)$ and let β be a root of $g(X)$. Let $\sigma \in \mathrm{Gal}(\bar{\mathbb{C}}_p/\mathbb{C}_p)$. Then $v(\alpha - \sigma\alpha) \geq \min\{v(\alpha - \beta), v(\sigma\alpha - \beta)\} = v(\alpha - \beta)$. If $\sigma \neq 1$, then we get an upper bound of $v(\alpha - \beta)$ that only depends on α. Let β_i be the roots of g, then $v(g(\alpha)) = \sum v(\alpha - \beta_i)$ has an upper bound. But $v(g(\alpha)) = v(g(\alpha) - f(\alpha))$ can be made arbitrarily large if we choose g to be close to f. The argument in this paragraph implicitly proves what is called Krasner's Lemma. $\square$

2. The theorem of Tate and Ax

The main reference for the following material is the paper *Zeros of Polynomials over Local Fields - The Galois Action* by James Ax, 1969.

The Galois group $G = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on $\mathbb{C}_p$ by isometries. Let H be a closed subgroup of G. We want to determine the fixed field $\mathbb{C}_p^H$. Of course $K := (\bar{\mathbb{Q}}_p)^H \subset \mathbb{C}_p^H$, and also $\hat{K} \subset \mathbb{C}_p^H$ because H acts continuously on $\mathbb{C}_p$. Here $\hat{K}$ is the completion (closure) of K inside $\mathbb{C}_p$.

Theorem 2.1 (Tate-Ax)**.** $\mathbb{C}_p^H = \hat{K}$.

Proof. Let $x \in \mathbb{C}_p^H$. Without loss of generality we may assume $v(x) \geq 0$. Let $\{x_n\} \subset \bar{\mathbb{Q}}_p$ be a sequence converging to x. We may assume $v(x_n - x) > n$. For $g \in H$ we have

$$v(gx_n - x_n) = v(gx_n - x + x - x_n) \geq \min\{v(gx_n - x), v(x_n - x)\} = v(x_n - x) > n.$$

By the following proposition, this implies that there exists $y_n \in K$ such that $v(y_n - x_n) \geq n - p/(p-1)^2$. Then we have $y_n \to \alpha$. $\square$

Proposition 2.2. *Let K be an algebraic extension of $\mathbb{Q}_p$. Let v be the p-adic valuation on K normalized by $v(p) = 1$. Let $x \in \bar{K}$ be such that for all $g \in G_K = \mathrm{Gal}(\bar{K}/K)$, $v(gx - x) \geq n$. Then there exists $y \in K$ such that $v(x - y) \geq n - p/(p-1)^2$. Simply put, if a small ball of $\bar{K}$ contains a whole Galois orbit, then by enlarging the small ball by a constant scalar, we get a ball that contains an element of K.*

We need the following lemma to prove the proposition.

Lemma 2.3. *Let $f(X) \in \bar{K}[X]$ be a monic polynomial of degree $d > 1$. If d is not a power of p, let q be the p-part of d. If d is a power of p, let $q = d/p$. Suppose $D = \{x | v(x - x_0) \geq \lambda\} \subset \bar{K}$ is a ball containing all the roots of f.*

(1) If d is not a p power, D contains a root of $f^{[q]} := f^{(q)}/q!$.
(2) If d is a p power, let $D' = \{x | v(x - x_0) \geq \lambda - 1/(d-q)\}$, an enlargement of D. Then D' contains a root of $f^{[q]}$.

Remark 2.4. This is a p-adic analogue of Gauss' Theorem: If a ball in $\mathbb{C}$ contains all the roots of a polynomial f, then it contains all the roots of f'.

Proof of Lemma. Assume we are in case (1). Without loss of generality we may assume the ball D is centered around $x_0 = 0$. Write $f(X) = \sum_{i=0}^{n} a_i X^i$. Then $f^{[q]}(0) = a_q$, so $v(f^{[q]}(0)) = v(a_q) \geq (d-q)\lambda$. Let β_i be the roots of $f^{[q]}$, then

$$f^{[q]}(0) = \binom{d}{q} \prod_{i=1}^{d-q} \beta_i$$

since the leading coefficient of $f^{[q]}$ is $\binom{d}{q}$. But $v(\binom{d}{q}) = 0$, so there is some β_i for which $v(\beta_i) \geq \lambda$.

In case (2), the argument is the same, the only difference being that now $v(\binom{d}{q}) = 1$.[1] □

Proof of Proposition. We prove the following statement by induction on the degree d of x.

Statement: If $x \in \bar{K}$ is such that for all $g \in G_K$, $v(x - gx) \geq n$, then there exists $y \in K$ such that

$$v(x-y) \geq n - \sum_{i=1}^{[\log_p d]} \frac{1}{p^i - p^{i-1}}.$$

Note that this inequality is stronger than that asserted in the Proposition. Let $f(X)$ be the monic minimal polynomial of x over K, of degree d.

For $d = 1$ we can take $y = x$. For the induction step, let $d > 1$. Fist suppose d is not a p power. Let q be the p-part of d. By Lemma, $f^{[q]}$ has a root β satisfying $v(x - \beta) \geq n$. For any $g \in G_K$, we have

$$v(\beta - g\beta) \geq \min\{v(\beta - x), v(x - gx), v(gx - g\beta)\} \geq n.$$

Let $d(\beta)$ be the degree of β, then $d(\beta) \leq d - q$. By induction hypothesis, there is an element $y \in K$ with

$$v(\beta - y) \geq n - \sum_{i=1}^{[\log_p d(\beta)]} \frac{1}{p^i - p^{i-1}}.$$

But $v(x-y) \geq \min\{v(x-\beta), v(\beta - y)\}$. So the statement is true for d.

Suppose d is a p power. Let $q = d/p$. Then by Lemma $f^{[q]}$ has a root β satisfying $v(x - \beta) \geq n - 1/(d-q)$. As before we see that for all $g \in G_K$,

$$v(g\beta - \beta) \geq n - 1/(d-q).$$

By induction hypothesis we find a $y \in K$ such that

$$v(\beta - y) \geq n - 1/(d-q) - \sum_{i=1}^{[\log_p d(\beta)]} \frac{1}{p^i - p^{i-1}}.$$

Then

$$v(x-y) \geq n - 1/(d-q) - \sum_{i=1}^{[\log_p d(\beta)]} \frac{1}{p^i - p^{i-1}} \geq n - \sum_{i=1}^{[\log_p d]} \frac{1}{p^i - p^{i-1}}.$$

□

[1] In the two cases when computing $v(\binom{d}{q})$, simply use the formula $v(n!) = \sum_i [n/p^i]$.

Remark 2.5. In the above proof, the element y that we found is in fact the root of the linear polynomial $f^{(d-1)}$, as can be seen from the induction process. This is none other than the arithmetic average of the conjugates of x. However if one makes the naive estimate $v(y) = v(\sum_g gx - x) - v(d) \geq \min_g v(gx - x) - v(d) \geq n - v(d)$, the lower bound depends on the degree d of x, which is not good enough for our purpose.